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SURVIVABILITY; A MARKOV PROCESS

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Abstract

Previous articles in this series have covered the time behavior of force strengths, battle trajectories, and recursive relationships between battle parameters. In this edition, the general solution to the Markov equation is derived and effects of segmented battle are compared with traditional conflict modes. Survivability, lethality, and repair are compared as command options. New graphic techniques are explored to reveal fundamental features of this combat structure.

INTRODUCTION

A Markov process is one in which the change between states is conservative and where the final state depends only on the initial state and a transition operator. In order to characterize this process, a 3X3 matrix representation is used to describe the operator. A state vector is also used to specify the current values of Blue, Red units (or tanks) and D (number of tanks of either side that have been damaged at that stage of the engagement). Once the operator and the initial state vector have been specified, the final or terminal state of combat can be derived.

The form for the operator and state vector product is

$$\begin{bmatrix} S_B & 0 & G_B \\ 0 & S_R & G_R \\ 1-S_B & 1-S_R & K \end{bmatrix}\begin{bmatrix} B \\ R \\ D \end{bmatrix}$$

Where

S_B = fraction of Blue tanks surviving the previous engagement

G_B = fraction of damaged tanks (both Blue and Red) put back in service on the Blue side.

$K = 1 - G_B - G_R$

B = current number of active Blue Tanks

R = current number of active Red tanks

In the material that follows the total number of tanks involved in the combat will be assumed to be 100. The state variables form a conservative system, i.e.,

$$B + R + D = 100$$

As the battle progresses between Blue and Red forces, a final eigenstate is reached that can be described by the following equations.

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$$S_B B_o + G_B D_o = B_o$$
$$S_R R_o + G_R D_o = R_o$$
$$(1 - S_B)B_o + (1 - S_R)R_o + KD_o = D_o$$

Solving these simultaneous equations yields the eigenstate components of the transition (or Markov) operator parameters.

$$B = 100\left(\frac{G_B}{1-S_B}\right)\left[\frac{G_B}{1-S_B} + \frac{G_R}{1-S_R} + 1\right]^{-1}$$
$$R = 100\left(\frac{G_R}{1-S_R}\right)\left[\frac{G_B}{1-S_B} + \frac{G_R}{1-S_R} + 1\right]^{-1}$$
$$D = 100\left[\frac{G_B}{1-S_B} + \frac{G_R}{1-S_R} + 1\right]^{-1}$$

THE GENERAL SOLUTION

Application of the Markov matrix or operator on the state vector yields the system trajectory.

$$\begin{bmatrix} S_B & 0 & G_B \\ 0 & S_R & G_R \\ 1-S_B & 1-S_R & K \end{bmatrix}\begin{bmatrix} B_n \\ R_n \\ D_n \end{bmatrix} = \begin{bmatrix} B_{n+1} \\ R_{n+1} \\ D_{n+1} \end{bmatrix}$$

Explicitly

$$B_{n+1} = S_B B_n + G_B D_n$$
$$R_{n+1} = S_R R_n + G_R D_n$$

During the operation of the matrix operator the components of the state variable are conserved, i.e.,

$$B_n + R_n + D_n = C$$

and expression for B_{n+1} can be written as

$$B_{n+1} = S_B B_n + G_B(C - B_n - R_n)$$
$$= (S_B - G_B)B_n - G_B R_n + G_B C$$

Subtracting B from both sides of the last equation yields

$$DB = B_{n+1} - B_n = (S_B - G_B - 1) - G_B R_n + G_B C$$
$$DR = (S_R - G_R - 1)R_n - G_R B_n + G_R C$$

In order to find the trajectory we need to solve the coupled differential equations

$$DX = A_x \bullet X - G_B \bullet Y + G_B C$$
$$DY = A_y \bullet Y - G_R \bullet X + G_R C$$

Where

$$X = B$$
$$Y = R$$
$$A_x = S_B - G_B - 1$$
$$A_y = S_R - G_R - 1$$

The equations involving DX and DY can be rearranged as follows

$$(D - A_x)X + G_B Y = G_B C$$
$$(D - A_y)Y + G_R X = G_R C$$

Treating $(D\text{-}A_y)$ as an operator yields the following pair of equations

$$(D - A_y)(D - A_x)x + G(D - A_y)y = -A_y G_B C$$

$$G_B G_R x + G_B(D - A_y)y = G_B G_R C$$

Eliminating "y" terms produces the following expression

$$[D^2 - (A_x + A_y)D + (A_x A_y - G_x G_y)]x$$

$$= -G_B C(A_y - G_R)$$

Expansion of this last equation yields

$$\left[D^2-(A_x+A_y)D+(A_xA_y-G_BG_R)\right]x$$
$$=-G_BC(A_y-G_R)$$

The general solution is as follows

$$x=C_1e^{m_+t}+C_2e^{m_-t}+W$$

where

$$m_\pm=\tfrac{1}{2}\left[\left(A_x+A_y\right)\pm\sqrt{\left(A_x-A_y\right)^2+4G_BG_R}\right]$$

and

$$W=-G_BC(A_y+G_R)/(A_xA_y-G_BG_R)$$

W is the final state value of x and can be written in the more familiar form

$$x_{final}=W=C\left(\frac{G_B}{1-S_B}\right)/\left(\frac{G_B}{1-S_B}+\frac{G_R}{1-S_R}+1\right)$$

SURVIVABILITY & LETHALITY

The question often asked is "what is more important; survivability or lethality? In order to examine this question, the equation for B_O can be rearranged to highlight S_B and $1-S_R$

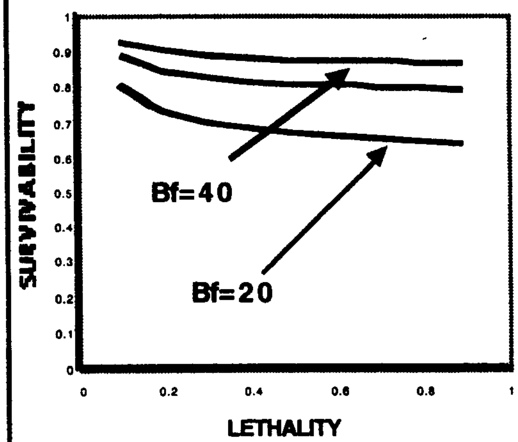
$$1-S_B=\left[\frac{10^2}{B_f}-1\right]\bullet\left[\frac{1-S_R}{1+10(1-S_R)}\right]$$

In the above equation $G_B=G_R=0.1$.

Figure (1) displays Blue force survivability versus Blue lethality where Blue's final force value is used as a parameter

FIGURE (1)

SURVIVABILITY VS LETHALITY



For high values of Blue survivability and significant final state values, lethality variations have little effect on the outcome of the battle. For instance, if the blue commander wants to have a final state of 40 tanks, choosing survivability values between 0.85 and 0.9 corresponds to having a lethality between 0.2 and 0.9. In other words, possessing high survivability allows considerable flexibility in using lethality values in achieving a given final combat vehicle number.

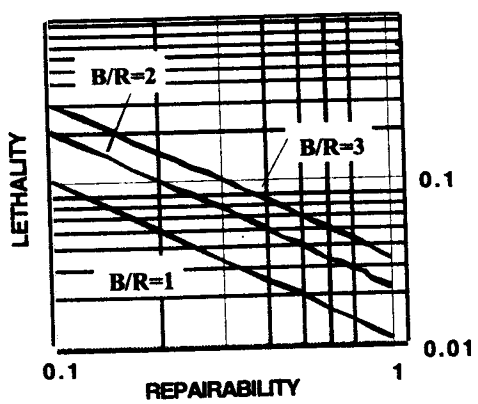
$1-S_R$	0.9			0.1		
B_f	40	30	20	40	30	20
R_f	6	7	8	30	33	72

Note that when Red survivability is high, the Red Force end strength can exceed Blue end strength. Final force strenghts are equal when survivability factors are equal.

LETHALITY & REPAIR

A second comparison involves Blue's lethality and repair. Assume that Blue's survival is 0.9 and Red's repair rate is 0.1. Figure (2) is a plot of $1\text{-}S_R$ (i.e. Blue's lethality) versus Blue's repair rate, G_B, for different final state ratios.

FIGURE 2
LETHALITY VS REPAIR



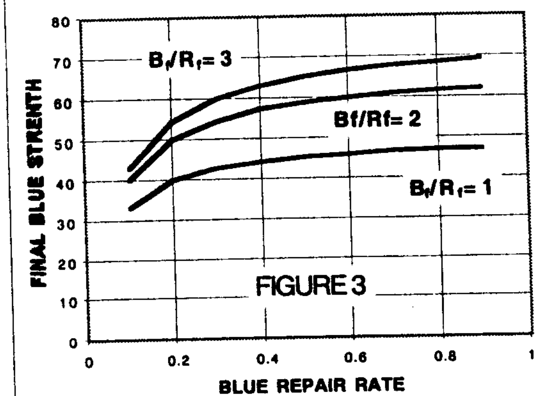
The combat equation for these results is

$$B_f / R_f = \frac{G_B(1-S_R)}{G_R(1-S_B)}$$

The Blue commander may wish to know if his remaining force strength will be sufficiently large. The expression

$$B_f = 100\left(\frac{G_B}{1-S_B}\right) \div \left(\frac{G_B}{1-S_B} + \frac{G_R}{1-S_R} + 1\right)$$

can then be used to find Blue's final strength. B_f versus G_B consistent with the values of G_R and $1\text{-}S_B$ used in Figure (2) are displayed in Figure (3).

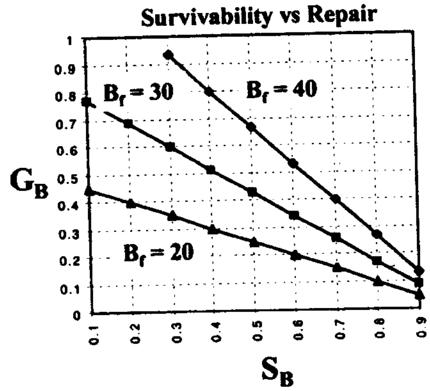


SURVIVABILITY & REPAIRABILITY

When Blue survival rates are compared with Blue repair rates (for constant values of B_f), Figure (4) is the result (In this example $S_R = 0.9$, $G_R = 0.1$, and $B_O = 50$).

Figure (4)

Survivability vs Repair



Lower values of S_B must be compensated by increased values of G_B in order to maintain constant B_f. For example, when $S_B = 0.4$, G_B must be 0.8 in order that B_f be equal to 40.

The defining equation in this instance is

$$S_B = 1 - G_B \frac{\left[\frac{100}{B_f} - 1\right]}{\left[\frac{G_R}{1-S_R}\right]}$$

It is useful to examine R_f values for the tree B_f values shown in Figure ().

B_f	40	30	20
R_f	30	35	40

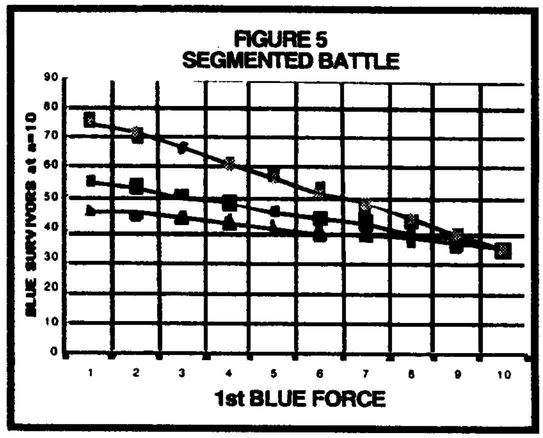
The diagonal $S_B = 1 - G_B$ in Figure divides the regions into victory and defeat regions.

$$S_B > 1 - G_B \xrightarrow{then} B_f > R_f$$
$$S_B < 1 - G_B \xrightarrow{then} B_f < R_f$$

SEGMENTED BATTLES

Markov type engagements are generally envisioned as combat encounters which are continuous from inception, or contact, until one side is obliterated or quits the battlefield. In real life, however, battles are frequently fought in stages ranging from skirmishes to major conflict.

Assume for the moment that the Markov engagement between Blue and Red forces only involves survivability factors of each force (which for this discussion are considered equal) and that the Blue commander first attacks a portion of the Red force until the nth stage of conflict. Starting at n+1, all the remaining Blue tanks engage all the surviving and not yet engaged Red units. Figure (5) shows the number of surviving Blue units at n=10 as a function of the number of Red targets attacked up and through the nth stage. The three curves in this figure represent various n values when full force engagement commences.



3D REPRESENTATIONS

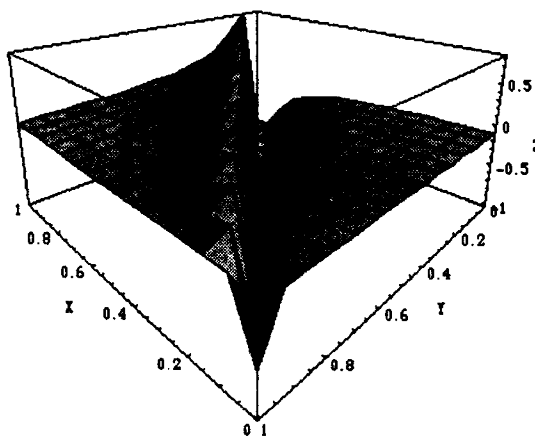


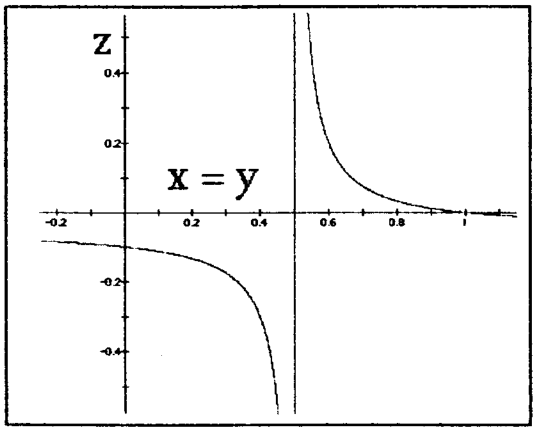
FIGURE 6:
SURVIVAL, LETHALITY, REPAIRABILITY

In Figure (6) the behavior of the Blue force is displayed in terms of three basic battle parameters; Blue survivability (y), Blue lethality (z), and Blue reparability(x). The corresponding equation of this surface is given by the following expression:

$$1 - S_R = G_R \left[\frac{G_B}{1 - S_B} \left(\frac{100}{B_f} - 1 \right) - 1 \right]^{-1}$$

When a plane described by the equation x=y is drawn in Figure (6), the intersection of the plane with the survival, lethality, reparability surfaces shown in Figure (7).

FIGURE 7
SLR CROSS SECTION



Lethality (Z) is always positive. In addition survivability and reparability are also positive parameters. The portion of Figure (7) that is positive in Z and lies beyond x = y = 0.5 is physically allowed. A negative value of lethality, while possible to interpret as enemy captures of Blue units, was not considered in this paper. When the repair rate for Red units is 0.1 and it is recognized that z cannot exceed a value of 1.0, examination of the last equation yields a lower limit of x (or y) of 0.524.

CONCLUSIONS

The first conclusion to be drawn from these calculations is that the Blue commander achieves superior benefit from investing in survivability compared with lethality improvements. In the example used in connection with Figure (1) changes in survivability were nearly fourteen times more effective than changes in lethality when the final Blue force strength is kept constant.

With respect to lethality and repair, consider the situation where Blue's repair rate is 0.6, final Blue force is 60 units, final Red force is 30 units, Blue Survivability is 0.9 and Red repair rate is 0.1. Then Equation () then indicates that Blue's lethality should be 0.0333. That same equation indicates that the product of repair rate and lethality should be a constant when the final states show a constant ratio. When Blue repair rate is 0.2, Figure (2) yields a Blue lethality value of 0.1 or 10 %.

Survivability and repair exhibit a linear relationship when Blue's final value is kept constant (see Figure 4)

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